

# Solution to a problem on hamiltonicity of graphs under Ore- and Fan-type heavy subgraph conditions\*

Bo Ning<sup>†</sup>, Shenggui Zhang<sup>‡</sup> and Binlong Li<sup>§</sup>

## Abstract

A graph  $G$  is called *claw- $o$ -heavy* if every induced claw ( $K_{1,3}$ ) of  $G$  has two end-vertices with degree sum at least  $|V(G)|$ . For a given graph  $S$ ,  $G$  is called  *$S$ - $f$ -heavy* if for every induced subgraph  $H$  of  $G$  isomorphic to  $S$  and every pair of vertices  $u, v \in V(H)$  with  $d_H(u, v) = 2$ , there holds  $\max\{d(u), d(v)\} \geq |V(G)|/2$ . In this paper, we prove that every 2-connected claw- $o$ -heavy and  $Z_3$ - $f$ -heavy graph is hamiltonian (with two exceptional graphs), where  $Z_3$  is the graph obtained by identifying one end-vertex of  $P_4$  (a path with 4 vertices) with one vertex of a triangle. This result gives a positive answer to a problem proposed in [B. Ning, S. Zhang, Ore- and Fan-type heavy subgraphs for Hamiltonicity of 2-connected graphs, Discrete Math. 313 (2013) 1715–1725], and also implies two previous theorems of Faudree et al. and Chen et al., respectively.

**Keywords:** Induced subgraphs; Claw- $o$ -heavy graphs;  $f$ -Heavy subgraphs; Hamiltonicity

**AMS Subject Classification (2000):** 05C38, 05C45

## 1 Introduction

Throughout this paper, the graphs considered are simple, finite and undirected. For terminology and notation not defined here, we refer the reader to Bondy and Murty [2].

Let  $G$  be a graph. For a vertex  $v \in V(G)$ , we use  $N_G(v)$  to denote the set, and  $d_G(v)$  the number, of neighbors of  $v$  in  $G$ . When there is no danger of ambiguity, we use  $N(v)$

---

\*Supported by NSFC (No. 11271300), and the project NEXLIZ – CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

<sup>†</sup>Center for Applied Mathematics, Tianjin University, Tianjin, 300072, P.R. China. Email: bo.ning@tju.edu.cn (B. Ning).

<sup>‡</sup>Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P.R. China. Email: sgzhang@nwpu.edu.cn (S. Zhang).

<sup>§</sup>Corresponding author. Northwestern Polytechnical University, Xi'an, Shaanxi 710072, P.R. China; Department of Mathematics, NTIS-New Technologies for the Information Society, University of West Bohemia, 30614 Pilsen, Czech Republic. Email: libinlong@mail.nwpu.edu.cn (B. Li).

and  $d(v)$  instead of  $N_G(v)$  and  $d_G(v)$ . If  $H$  and  $H'$  are two subgraphs of  $G$ , then we set  $N_H(H') = \{v \in V(H) : N_G(v) \cap V(H') \neq \emptyset\}$ . For two vertices  $u, v \in V(H)$ , the *distance* between  $u$  and  $v$  in  $H$ , denoted by  $d_H(u, v)$ , is the length of a shortest path connecting  $u$  and  $v$  in  $H$ . In particular, when we use the notation  $G$  to denote a graph, then for some subgraph  $H$  of  $G$ , we set  $N_H(v) = N_G(v) \cap V(H)$  and  $d_H(v) = |N_H(v)|$  (so, if  $G'$  is another graph defined on the same vertex set  $V(G)$  and  $H$  is a subgraph of  $G'$ , we will not use  $N_H(v)$  to denote  $N_{G'}(v) \cap V(H)$ ).

We call  $H$  an *induced subgraph* of  $G$ , if for every  $x, y \in V(H)$ ,  $xy \in E(G)$  implies that  $xy \in E(H)$ . For a given graph  $S$ ,  $G$  is called  *$S$ -free* if  $G$  contains no induced subgraph isomorphic to  $S$ . Following [8],  $G$  is called  *$S$ -o-heavy* if every induced subgraph of  $G$  isomorphic to  $S$  contains two nonadjacent vertices with degree sum at least  $|V(G)|$  in  $G$ . Following [9],  $G$  is called  *$S$ -f-heavy* if for every induced subgraph  $H$  isomorphic to  $S$  and any two vertices  $u, v \in V(H)$  such that  $d_H(u, v) = 2$ , there holds  $\max\{d(u), d(v)\} \geq |V(G)|/2$ . Note that an  $S$ -free graph is  $S$ -o-heavy ( $S$ -f-heavy).

The *claw* is the bipartite graph  $K_{1,3}$ . Note that a claw- $f$ -heavy graph is also claw- $o$ -heavy. Further graphs that will be often considered as forbidden subgraphs are shown in Fig. 1.

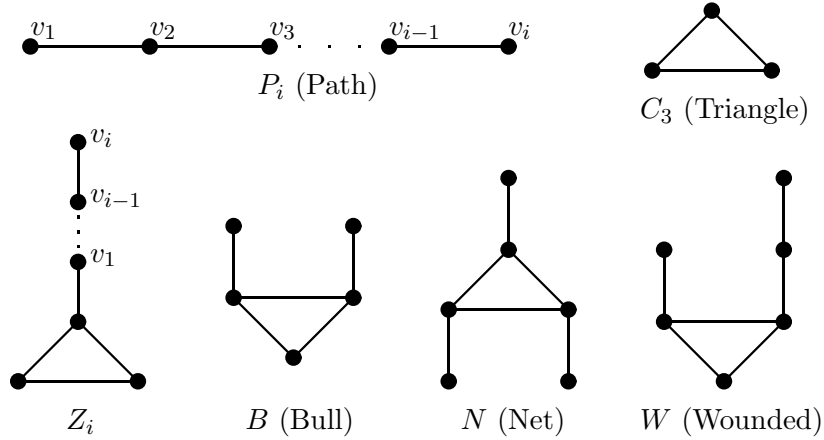


Fig. 1. Graphs  $P_i, C_3, Z_i, B, N$  and  $W$ .

Bedrossian [1] characterized all connected forbidden pairs for a 2-connected graph to be hamiltonian.

**Theorem 1.** (Bedrossian [1]) *Let  $G$  be a 2-connected graph and let  $R$  and  $S$  be connected graphs other than  $P_3$ . Then  $G$  being  $R$ -free and  $S$ -free implies  $G$  is hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S = C_3, P_4, P_5, P_6, Z_1, Z_2, B, N$  or  $W$ .*

Faudree and Gould [6] extended Bedrossian's result by giving a proof of the 'only if' part based on infinite families of non-hamiltonian graphs.

**Theorem 2.** (Faudree and Gould [6]) *Let  $G$  be a 2-connected graph of order at least 10 and let  $R$  and  $S$  be connected graphs other than  $P_3$ . Then  $G$  being  $R$ -free and  $S$ -free implies  $G$  is hamiltonian if and only if (up to symmetry)  $R = K_{1,3}$  and  $S = C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$  or  $W$ .*

Li et al. [8] extended Bedrossian's result by restricting Ore's condition to pairs of induced subgraphs of a graph. Ning and Zhang [9] gave another extension of Bedrossian's theorem by restricting Ore's condition to induced claws and Fan's condition to other induced subgraphs of a graph.

**Theorem 3.** (Ning and Zhang [9]) *Let  $G$  be a 2-connected graph and  $S$  be a connected graph other than  $P_3$ . Suppose that  $G$  is claw- $o$ -heavy. Then  $G$  being  $S$ - $f$ -heavy implies  $G$  is hamiltonian if and only if  $S = P_4, P_5, P_6, Z_1, Z_2, B, N$  or  $W$ .*

Motivated by Theorems 2 and 3, Ning and Zhang [9] proposed the following problem.

**Problem 1.** (Ning and Zhang [9]) *Is every claw- $o$ -heavy and  $Z_3$ - $f$ -heavy graph of order at least 10 hamiltonian?*

The main goal of this paper is to give an affirmative solution to this problem. Our answer is the following theorem, where the graphs  $L_1$  and  $L_2$  are shown in Fig. 2.

**Theorem 4.** *Let  $G$  be a 2-connected graph. If  $G$  is claw- $o$ -heavy and  $Z_3$ - $f$ -heavy, then  $G$  is either hamiltonian or isomorphic to  $L_1$  or  $L_2$ .*

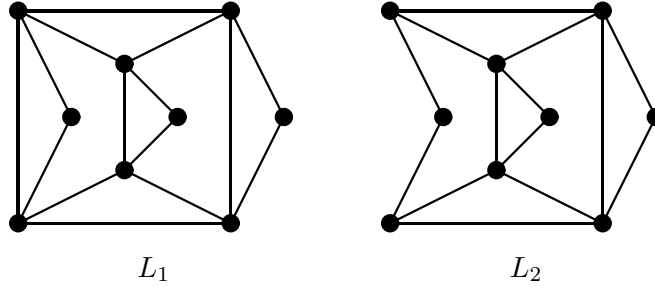


Fig. 2. Graphs  $L_1$  and  $L_2$ .

Theorem 4 extends the following two previous theorems.

**Theorem 5.** (Faudree et al. [7]) *If  $G$  is a 2-connected claw-free and  $Z_3$ -free graph, then  $G$  is either hamiltonian or isomorphic to  $L_1$  or  $L_2$ .*

**Theorem 6.** (Chen et al. [5]) *If  $G$  is a 2-connected claw- $f$ -heavy and  $Z_3$ - $f$ -heavy graph, then  $G$  is either hamiltonian or isomorphic to  $L_1$  or  $L_2$ .*

We remark that there are infinite 2-connected claw-*o*-heavy and  $Z_3$ -*o*-heavy graphs which are non-hamiltonian, see [8].

Together with Theorem 3 and Theorem 4, we can obtain the following result which generalizes Theorem 2.

**Theorem 7.** *Let  $G$  be a 2-connected graph of order at least 10 and  $S$  be a connected graph other than  $P_3$ . Suppose that  $G$  is claw-*o*-heavy. Then  $G$  being  $S$ -*f*-heavy implies  $G$  is hamiltonian if and only if  $S = P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$  or  $W$ .*

## 2 Preliminaries

In this section, we will list some necessary preliminaries. First, we will introduce the closure theory of claw-*o*-heavy graphs proposed by Čada [4], which is an extension of the closure theory of claw-free graphs due to Ryjáček [10].

Let  $G$  be a graph of order  $n$ . A vertex  $x \in V(G)$  is called *heavy* if  $d(x) \geq n/2$ ; otherwise, it is called *light*. A pair of nonadjacent vertices  $\{x, y\} \subset V(G)$  is called a *heavy pair* of  $G$  if  $d(x) + d(y) \geq n$ .

Let  $G$  be a graph and  $x \in V(G)$ . Define  $B_x^o(G) = \{uv : \{u, v\} \subset N(x), d(u) + d(v) \geq |V(G)|\}$ . Let  $G_x^o$  be a graph with vertex set  $V(G_x^o) = V(G)$  and edge set  $E(G_x^o) = E(G) \cup B_x^o(G)$ . Suppose that  $G_x^o[N(x)]$  consists of two disjoint cliques  $C_1$  and  $C_2$ . For a vertex  $y \in V(G) \setminus (N(x) \cup \{x\})$ , if  $\{x, y\}$  is a heavy pair in  $G$  and there are two vertices  $x_1 \in C_1$  and  $x_2 \in C_2$  such that  $x_1y, x_2y \in E(G)$ , then  $y$  is called a *join vertex* of  $x$  in  $G$ . If  $N(x)$  is not a clique and  $G_x^o[N(x)]$  is connected, or  $G_x^o[N(x)]$  consists of two disjoint cliques and there is some join vertex of  $x$ , then the vertex  $x$  is called an *o-eligible vertex* of  $G$ . The *locally completion of  $G$  at  $x$* , denoted by  $G'_x$ , is the graph with vertex set  $V(G'_x) = V(G)$  and edge set  $E(G'_x) = E(G) \cup \{uv : u, v \in N(x)\}$ .

Let  $G$  be a claw-*o*-heavy graph. The *closure* of  $G$ , denoted by  $cl_o(G)$ , is the graph such that:

- (1) there is a sequence of graphs  $G_1, G_2, \dots, G_t$  such that  $G = G_1$ ,  $G_t = cl_o(G)$ , and for any  $i \in \{1, 2, \dots, t-1\}$ , there is an *o*-eligible vertex  $x_i$  of  $G_i$ , such that  $G_{i+1} = (G_i)'_{x_i}$ ; and
- (2) there is no *o*-eligible vertex in  $G_t$ .

**Theorem 8.** (Čada [4]) *Let  $G$  be a claw-*o*-heavy graph. Then*

- (1) *the closure  $cl_o(G)$  is uniquely determined;*
- (2) *there is a  $C_3$ -free graph  $H$  such that  $cl_o(G)$  is the line graph of  $H$ ; and*
- (3) *the circumferences of  $cl_o(G)$  and  $G$  are equal.*

Now we introduce some new terminology and notations. Let  $G$  be a claw- $o$ -heavy graph and  $C$  be a maximal clique of  $cl_o(G)$ . We call  $G[C]$  a *region* of  $G$ . For a vertex  $v$  of  $G$ , we call  $v$  an *interior vertex* if it is contained in only one region, and a *frontier vertex* if it is contained in two distinct regions. For two vertices  $u, v \in V(G)$ , we say  $u$  and  $v$  are *associated* if  $u, v$  are contained in a common region of  $G$ ; otherwise  $u$  and  $v$  are *dissociated*. For a region  $R$  of  $G$ , we denote by  $I_R$  the set of interior vertices of  $R$ , and by  $F_R$  the set of frontier vertices of  $R$ .

From the definition of the closure, it is not difficult to get the following lemma.

**Lemma 1.** *Let  $G$  be a claw- $o$ -heavy graph. Then*

- (1) *every vertex is either an interior vertex of a region or a frontier vertex of two regions;*
- (2) *every two regions are either disjoint or have only one common vertex; and*
- (3) *every pair of dissociated vertices have degree sum less than  $|V(G)|$  in  $cl_o(G)$  (and in  $G$ ).*

*Proof.* In the proof of the lemma, we let  $G' = cl_o(G)$ .

(1) Let  $v$  be an arbitrary vertex of  $G$ . Since  $G'$  is closed,  $N_{G'}(v)$  is either a clique or a disjoint union of two cliques in  $G'$ . Thus  $v$  is contained in one or two regions of  $G$ , and the assertion is true.

(2) Let  $R$  and  $R'$  be two regions of  $G$ , and  $C$  and  $C'$  be the two maximal cliques of  $G'$  corresponding to  $R$  and  $R'$ , respectively. If  $C$  and  $C'$  have two common vertices, say  $u$  and  $v$ , then  $u$  and  $v$  will be  $o$ -eligible vertices of  $G'$ , contradicting the definition of the closure of  $G$ . This implies that  $C$  and  $C'$  (and then,  $R$  and  $R'$ ) have at most one common vertex.

(3) Let  $u, v$  be two nonadjacent vertices with  $d_{G'}(u) + d_{G'}(v) \geq n = |V(G)|$ . Then  $u, v$  have at least two common neighbors in  $G'$ . Suppose that  $u$  and  $v$  are not in a common clique of  $G'$ . Let  $x$  be a common neighbor of  $u$  and  $v$  in  $G'$ . Since  $N_{G'}(x)$  is not a clique in  $G'$ , it is the disjoint union of two cliques, one containing  $u$  and the other containing  $v$ . Since  $uv \in B_x^o(G')$ ,  $x$  is an  $o$ -eligible vertex of  $G'$ , a contradiction. Thus we conclude that  $u, v$  are in a common clique of  $G'$ , i.e.,  $u$  and  $v$  are associated.  $\square$   $\square$

The next lemma provides some structural information on regions.

**Lemma 2.** *Let  $G$  be a claw- $o$ -heavy graph and  $R$  be a region of  $G$ . Then*

- (1)  *$R$  is nonseparable;*
- (2) *if  $v$  is a frontier vertex of  $R$ , then  $v$  has an interior neighbor in  $R$  or  $R$  is complete and has no interior vertices;*
- (3) *for any two vertices  $u, v \in R$ , there is an induced path of  $G$  from  $u$  to  $v$  such that every*

internal vertex of the path is an interior vertex of  $R$ ; and

(4) for two vertices  $u, v$  in  $R$ , if  $\{u, v\}$  is a heavy pair of  $G$ , then  $u, v$  have two common neighbors in  $I_R$ .

*Proof.* Let  $G_1, G_2, \dots, G_t$  be the sequence of graphs, and  $x_1, x_2, \dots, x_{t-1}$  the sequence of vertices in the definition of  $cl_o(G)$ .

(1) Suppose that  $R$  has a cut-vertex  $y$ . We prove by induction that  $y$  would be a cut-vertex of  $G_i[V(R)]$  for all  $i \in [1, t]$ . Since  $y$  is a cut-vertex of  $G_1[V(R)] = R$ , we assume that  $2 \leq i \leq t$ . By the induction hypothesis,  $y$  is a cut-vertex of  $G_{i-1}[V(R)]$ . Let  $R'$  and  $R''$  be two components of  $G_{i-1}[V(R)] - y$ ,  $u$  be a vertex of  $R'$  and  $v$  be a vertex of  $R''$ . Then  $u$  and  $v$  have at most one common neighbor  $y$  in  $R$ . Note that each two maximal cliques of  $cl_o(G)$  is either disjoint or have only one common vertex (see Lemma 1 (1)). This implies that  $u$  and  $v$  have no common neighbors in  $G_{i-1} - V(R)$ . Hence  $\{u, v\}$  is not a heavy pair of  $G$ . Note that an  $o$ -eligible vertex of  $G_{i-1}$  will be an interior vertex of  $cl_o(G)$ . This implies that  $y$  is not an  $o$ -eligible vertex of  $G_{i-1}$ . Thus  $x_{i-1} \neq y$ . Note that  $x_{i-1}$  has no neighbors in  $R'$  or has no neighbors in  $R''$ . This implies that there are no new edges in  $G_i$  between  $R'$  and  $R''$ . Thus  $y$  is also a cut-vertex of  $G_i[V(R)]$ . By induction, we can see that  $y$  is a cut-vertex of  $cl_o(G)[V(R)]$ , contradicting the fact that  $V(R)$  is a clique in  $cl_o(G)$ .

(2) Note that  $cl_o(G)[V(R)]$  is complete. If  $R$  has no interior vertex, then  $R$  contains no  $o$ -eligible vertex of  $G$ . Since the locally completion of  $G$  at every  $o$ -eligible vertex does not add an edge in  $R$ ,  $R = cl_o(G)[V(R)]$  is complete.

Now we assume that  $R$  has at least one interior vertex. Suppose that  $v$  has no interior neighbors in  $R$ , i.e.,  $N(v) \cap I_R = \emptyset$ . Using induction, we will prove that  $N_{G_i}(v) \cap I_R = \emptyset$ . Since  $N_{G_1}(v) \cap I_R = \emptyset$ , we assume that  $2 \leq i \leq t$ . By the induction hypothesis,  $N_{G_{i-1}}(v) \cap I_R = \emptyset$ . Note that  $x_{i-1}$  is either nonadjacent to  $v$  or nonadjacent to every vertex in  $N_{G_{i-1}}(v) \cap V(R)$ . This implies that there are no new edges of  $G_i$  between  $v$  and  $G_i[V(R)] - v$ . Hence  $N_{G_i}(v) \cap I_R = \emptyset$ . Thus by the induction hypothesis, we can see that  $N_{cl_o(G)}(v) \cap I_R = \emptyset$ , a contradiction.

(3) We use induction on  $t - i$  ( $t$  is the subscript of  $G_t = cl_o(G)$ ) to prove that there is an induced path of  $G_i[V(R)]$  from  $u$  to  $v$  such that every internal vertex of the path is an interior vertex of  $R$ . Note that  $uv$  is an edge in  $G_t[V(R)]$ . We are done if  $i = t$ . Now suppose that there is an induced path  $P$  of  $G_i[V(R)]$  from  $u$  to  $v$  such that every internal vertex of the path is an interior vertex of  $R$ . We will prove that there is an induced path of  $G_{i-1}[V(R)]$  from  $u$  to  $v$  such that every internal vertex of the path is an interior vertex of  $R$ . If  $P$  is also a path of  $G_{i-1}[V(R)]$ , then we are done. So we assume that there is an

edge  $u'v' \in E(P)$  such that  $u'v' \notin E(G_{i-1})$ . This implies that  $u', v' \in N(x_{i-1})$ . Since  $P$  is an induced path of  $G_i$ ,  $x_{i-1}$  has the only two neighbors  $u', v'$  on  $P$ . We also note that  $x_{i-1} \in V(R)$  is an interior vertex. Thus  $P' = (P - u'v') \cup u'xv'$  (with the obvious meaning) is an induced path of  $G_{i-1}[V(R)]$  from  $u$  to  $v$  such that every internal vertex of the path is an interior vertex of  $R$ . Thus by the induction hypothesis, the proof is complete.

(4) Since every vertex in  $F_R$  has at least one neighbor in  $G - R$  and every vertex in  $G - R$  has at most one neighbor in  $F_R$ , we have  $|N_{G-R}(F_R \setminus \{u, v\})| \geq |F_R \setminus \{u, v\}|$ . Furthermore, we have  $n = |I_R \setminus \{u, v\}| + |F_R \setminus \{u, v\}| + |V(G - R)| + 2$ . Thus, we get

$$\begin{aligned}
n &\leq d(u) + d(v) \\
&= d_{I_R}(u) + d_{I_R}(v) + d_{F_R}(u) + d_{F_R}(v) + d_{G-R}(u) + d_{G-R}(v) \\
&\leq d_{I_R}(u) + d_{I_R}(v) + 2|F_R \setminus \{u, v\}| + d_{G-R}(u) + d_{G-R}(v) \\
&\leq d_{I_R}(u) + d_{I_R}(v) + |F_R \setminus \{u, v\}| + |N_{G-R}(F_R \setminus \{u, v\})| + |N_{G-R}(u)| + |N_{G-R}(v)| \\
&= d_{I_R}(u) + d_{I_R}(v) + |F_R \setminus \{u, v\}| + |N_{G-R}(F_R)| \\
&\leq d_{I_R}(u) + d_{I_R}(v) + |F_R \setminus \{u, v\}| + |V(G - R)|,
\end{aligned}$$

and

$$d_{I_R}(u) + d_{I_R}(v) \geq n - |F_R \setminus \{u, v\}| - |V(G - R)| = |I_R \setminus \{u, v\}| + 2.$$

This implies that  $u, v$  have two common neighbors in  $I_R$ . □ □

Let  $G$  be a graph and  $Z$  be an induced copy of  $Z_3$  in  $G$ . We denote the vertices of  $Z$  as in Fig. 3, and say that  $Z$  is *center-heavy* in  $G$  if  $a_1$  is a heavy vertex of  $G$ . If every induced copy of  $Z_3$  in  $G$  is center-heavy, then we say that  $G$  is  *$Z_3$ -center-heavy*.

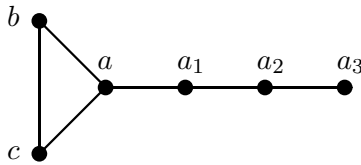


Fig. 3. The Graph  $Z_3$ .

**Lemma 3.** *Let  $G$  be a claw-o-heavy and  $Z_3$ -f-heavy graph. Then  $cl_o(G)$  is  $Z_3$ -center-heavy.*

*Proof.* Let  $Z$  be an arbitrary induced copy of  $Z_3$  in  $G' = cl_o(G)$ . We denote the vertices of  $Z$  as in Fig. 3, and will prove that  $a_1$  is heavy in  $G'$ .

Let  $R$  be the region of  $G$  containing  $\{a, b, c\}$ . Recall that  $I_R$  is the set of interior vertices of  $R$ , and  $F_R$  is the set of frontier vertices of  $R$ .

**Claim 1.**  $|N_R(a_2) \cup N_R(a_3)| \leq 1$ .

*Proof.* Note that every vertex in  $G - R$  has at most one neighbor in  $R$ . If  $N_R(a_2) = \emptyset$ , then the assertion is obviously true. Now we assume that  $N_R(a_2) \neq \emptyset$ . Let  $x$  be the vertex in  $N_R(a_2)$ . Clearly  $x \neq a$  and  $a_1x \notin E(G')$ . If  $a_3x \notin E(G')$ , then  $\{a_2, a_1, a_3, x\}$  induces a claw in  $G'$ , a contradiction. This implies that  $a_3x \in E(G')$ , and  $x$  is the unique vertex in  $N_{G'}(a_3) \cap V(R)$ . Thus  $N_R(a_2) \cup N_R(a_3) = \{x\}$ .  $\square$   $\square$

**Claim 2.** Let  $x, y$  be two vertices in  $I_R \cup \{a\}$ . If  $xy \in E(G)$  and  $d(x) + d(y) \geq n$ , then  $x, y$  have a common neighbor in  $I_R$ .

*Proof.* Note that every vertex in  $F_R$  has at least one neighbor in  $G - R$  and every vertex in  $G - R$  has at most one neighbor in  $R$ . By Claim 1,  $|V(G - R)| \geq |F_R| + 1$ . Moreover, since  $a$  is not the neighbor of  $a_2$  and  $a_3$  in  $R$ ,  $|V(G - R)| \geq |F_R \setminus \{a\}| + |N_{G-R}(a)| + 1$ .

If  $x, y \in I_R$ , then

$$\begin{aligned} n &\leq d(x) + d(y) \\ &= d_{I_R}(x) + d_{I_R}(y) + d_{F_R}(x) + d_{F_R}(y) \\ &\leq d_{I_R}(x) + d_{I_R}(y) + 2|F_R| \\ &\leq d_{I_R}(x) + d_{I_R}(y) + |F_R| + |V(G - R)| - 1, \end{aligned}$$

and

$$d_{I_R}(x) + d_{I_R}(y) \geq n - |F_R| - |V(G - R)| + 1 = |I_R| + 1.$$

This implies that  $x, y$  have a common neighbor in  $I_R$ .

If one of  $x, y$ , say  $y$  is  $a$ , then

$$\begin{aligned} n &\leq d(x) + d(a) \\ &= d_{I_R}(x) + d_{I_R}(a) + d_{F_R}(x) + d_{F_R}(a) + d_{G-R}(a) \\ &\leq d_{I_R}(x) + d_{I_R}(a) + |F_R| + |F_R \setminus \{a\}| + d_{G-R}(a) \\ &\leq d_{I_R}(x) + d_{I_R}(a) + |F_R| + |V(G - R)| - 1, \end{aligned}$$

and

$$d_{I_R}(x) + d_{I_R}(a) \geq n - |F_R| - |V(G - R)| + 1 = |I_R| + 1.$$

This implies that  $x, a$  have a common neighbor in  $I_R$ .  $\square$   $\square$

By Lemma 2 (3),  $G$  has an induced path  $P$  from  $a$  to  $a_3$  such that every vertex of  $P$  is either in  $\{a, a_1, a_2, a_3\}$  or an interior vertex outside  $R$ . Let  $a, a'_1, a'_2, a'_3$  be the first four vertices of  $P$ .



Note that  $a'_1$  is either  $a_1$  or an interior vertex in the region containing  $\{a, a_1\}$ . This implies that  $d_{G'}(a_1) \geq d_{G'}(a'_1) \geq d(a'_1)$ . If  $a'_1$  is heavy in  $G$ , then  $a_1$  is heavy in  $G'$  and we are done. So we assume that  $a'_1$  is not heavy in  $G$ .

If  $abca$  is also a triangle in  $G$ , then the subgraph induced by  $\{a, b, c, a'_1, a'_2, a'_3\}$  is a  $Z_3$ . Since  $G$  is  $Z_3$ - $f$ -heavy and  $a'_1$  is not heavy in  $G$ ,  $b$  and  $a'_3$  are heavy in  $G$ . By Lemma 1 (3),  $b$  and  $a'_3$  are associated, a contradiction. Thus we conclude that one edge of  $\{ab, ac, bc\}$  is not in  $E(G)$ .

Note that  $R$  is not complete. By Lemma 2 (2),  $a$  has a neighbor in  $I_R$ .

**Claim 3.**  $d_{I_R}(a) = 1$ .

*Proof.* Suppose that  $d_{I_R}(a) \geq 2$ . Let  $x, y$  be two arbitrary vertices in  $N_{I_R}(a)$ . If  $xy \in E(G)$ , then  $\{a, x, y, a'_1, a'_2, a'_3\}$  induces a  $Z_3$  in  $G$ . Note that  $a'_1$  is not heavy in  $G$ . Thus  $x$  and  $a'_3$  are heavy in  $G$ . Note that  $x$  and  $a'_3$  are dissociated, a contradiction. This implies that  $N_{I_R}(a)$  is an independent set.

Since  $\{a, x, y, a'_1\}$  induces a claw in  $G$ , and  $\{a'_1, x\}, \{a'_1, y\}$  are not heavy pairs of  $G$  by Lemma 1 (3), we have  $\{x, y\}$  is a heavy pair of  $G$ . We assume without loss of generality that  $x$  is heavy in  $G$ .

If  $a$  is also heavy in  $G$ , then by Claim 2,  $a, x$  have a common neighbor in  $I_R$ , contradicting the fact that  $N_{I_R}(a)$  is an independent set. So we conclude that  $a$  is not heavy in  $G$ .

Since  $\{x, y\}$  is a heavy pair of  $G$ , by Lemma 2 (4),  $x, y$  have two common neighbors in  $I_R$ . Let  $x', y'$  be two vertices in  $N_{I_R}(x) \cap N_{I_R}(y)$ . Clearly  $ax', ay' \notin E(G)$ . If  $x'y' \in E(G)$ , then  $\{x, x', y', a, a'_1, a'_2\}$  induces a  $Z_3$  in  $G$ . Since  $a$  is light,  $x', a'_2$  are heavy. Note that  $x'$  and  $a'_2$  are dissociated, a contradiction. Thus we obtain that  $x'y' \notin E(G)$ .

Note that  $\{x, x', y', a\}$  induces a claw in  $G$ , and  $a$  is light in  $G$ . So one vertex of  $\{x', y'\}$ , say  $x'$ , is heavy in  $G$ . By Claim 2,  $x, x'$  have a common neighbor  $x''$  in  $I_R$ . Clearly  $ax'' \notin E(G)$ . Thus  $\{x, x', x'', a, a'_1, a'_2\}$  induces a  $Z_3$ . Since  $a$  is not heavy in  $G$ ,  $x', a'_2$  are heavy in  $G$ , a contradiction.  $\square$   $\square$

Now let  $N_{I_R}(a) = \{x\}$ .

**Claim 4.**  $N_R(a) = V(R) \setminus \{a\}$ .

*Proof.* Suppose that  $V(R) \setminus (\{a\} \cup N_R(a)) \neq \emptyset$ . By Lemma 2 (1),  $R - x$  is connected. Let  $y$  be a vertex in  $V(R) \setminus (\{a\} \cup N_R(a))$  such that  $a, y$  have a common neighbor  $z$  in  $R - x$ . Note that  $z$  is a frontier vertex of  $R$ . Let  $z'$  be a vertex in  $N_{G-R}(z)$ . Then  $\{z, y, a, z'\}$  induces a claw in  $G$ . Since  $\{a, z'\}, \{y, z'\}$  are not heavy pairs of  $G$ ,  $\{a, y\}$  is a heavy

pair of  $G$ . By Lemma 2 (4),  $a, y$  have two common neighbors in  $I_R$ , contradicting Claim 3.  $\square$   $\square$

By Claims 3 and 4, we can see that  $|I_R| = 1$ . Recall that one edge of  $\{ab, bc, ac\}$  is not in  $E(G)$ . By Claim 4,  $ab, ac \in E(G)$ . This implies that  $bc \notin E(G)$ , and  $\{a, b, c, a'_1\}$  induces a claw in  $G$ . Since  $\{b, a'_1\}, \{c, a'_1\}$  are not heavy pairs of  $G$ ,  $\{b, c\}$  is a heavy pair of  $G$ . By Lemma 2 (4),  $b$  and  $c$  have two common neighbors in  $I_R$ , contradicting the fact that  $|I_R| = 1$ .  $\square$   $\square$

Following [3], we define  $\mathcal{P}$  to be the class of graphs obtained by taking two vertex-disjoint triangles  $a_1a_2a_3a_1$ ,  $b_1b_2b_3b_1$  and by joining every pair of vertices  $\{a_i, b_i\}$  by a path  $P_{k_i} = a_i c_i^1 c_i^2 \cdots c_i^{k_i-2} b_i$ , for  $k_i \geq 3$  or by a triangle  $a_i b_i c_i a_i$ . We denote the graphs in  $\mathcal{P}$  by  $P_{l_1, l_2, l_3}$ , where  $l_i = k_i$  if  $a_i, b_i$  are joined by a path  $P_{k_i}$ , and  $l_i = T$  if  $a_i, b_i$  are joined by a triangle. Note that  $L_1 = P_{T, T, T}$  and  $L_2 = P_{3, T, T}$ .

**Theorem 9.** (Brousek [3]) *Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph  $H \in \mathcal{P}$ .*

### 3 Proof of Theorem 4

Let  $G' = cl_o(G)$ . If  $G'$  is hamiltonian, then so is  $G$  by Theorem 8, and we are done. Now we assume that  $G'$  is not hamiltonian. By Theorem 9,  $G'$  contains an induced subgraph  $H = P_{l_1, l_2, l_3} \in \mathcal{P}$ . We denote the vertices of  $H$  by  $a_i, b_i, c_i$  and  $c_i^j$  as in Section 2. By Lemma 3,  $G'$  is  $Z_3$ -center-heavy.

**Claim 1.** For  $i \in \{1, 2, 3\}$ ,  $l_i = 3$  or  $T$ ; and at most one of  $\{l_1, l_2, l_3\}$  is 3.

*Proof.* If one of  $\{l_1, l_2, l_3\}$  is at least 4, say  $l_1 \geq 4$ , then the subgraph of  $G'$  induced by  $\{a_1, a_2, a_3, c_1^1, c_1^2, c_1^3\}$  is a  $Z_3$  (we set  $c_1^3 = b_1$  if  $l_1 = 4$ ). Thus  $c_1^1$  is heavy in  $G'$ . If  $l_2 = T$ , then the subgraph of  $G'$  induced by  $\{a_2, a_1, a_3, b_2, b_1, c_1^{l_1-2}\}$  is a  $Z_3$ , implying  $b_2$  is heavy in  $G'$ . But  $c_1^1$  and  $b_2$  are dissociated, a contradiction. If  $l_2 \neq T$ , then the subgraph of  $G'$  induced by  $\{a_2, a_1, a_3, c_2^1, \dots, c_2^{l_2-2}, b_2, b_1\}$  is a  $Z_r$  with  $r \geq 3$ , implying  $c_2^1$  is heavy in  $G'$ . But  $c_1^1$  and  $c_2^1$  are dissociated, a contradiction again. Thus we conclude that  $l_i = 3$  or  $T$  for all  $i = 1, 2, 3$ .

If two of  $\{l_1, l_2, l_3\}$  equal 3, say  $l_1 = l_2 = 3$ , then the subgraphs of  $G'$  induced by  $\{a_1, a_2, a_3, c_1^1, b_1, b_2\}$  and by  $\{a_2, a_1, a_3, c_2^1, b_2, b_1\}$  are  $Z_3$ 's. This implies that  $c_1^1$  and  $c_2^1$  are heavy in  $G'$ . But  $c_1^1$  and  $c_2^1$  are dissociated, a contradiction. Thus we conclude that at most one of  $\{l_1, l_2, l_3\}$  is 3.  $\square$   $\square$

By Claim 1, we assume without loss of generality that  $l_2 = l_3 = T$  and  $l_1 = 3$  or  $T$ . If  $G'$  has only the nine vertices in  $H$ , then  $G' = L_1$  or  $L_2$ , and  $G$  has no  $o$ -eligible vertices. This implies that  $G = L_1$  or  $L_2$ . Now we assume that  $G'$  has a tenth vertex.

Let  $A$  be the region containing  $\{a_1, a_2, a_3\}$  and  $B$  be the region containing  $\{b_1, b_2, b_3\}$ . For  $l_i = T$ , let  $C_i$  be the region containing  $\{a_i, b_i, c_i\}$ ; and if  $l_1 = 3$ , then let  $C_1^1$  and  $C_1^2$  be the regions containing  $\{a_1, c_1^1\}$  and  $\{b_1, c_1^1\}$ , respectively.

**Claim 2.**  $|V(A)| = |V(B)| = |V(C_i)| = 3$ ; and if  $l_1 = 3$ , then  $|V(C_1^1)| = |V(C_1^2)| = 2$ .

*Proof.* Suppose that  $|V(A)| \geq 4$ . Let  $x$  be a vertex in  $V(A) \setminus \{a_1, a_2, a_3\}$ . Then the subgraphs of  $G'$  induced by  $\{a_2, a_1, x, b_2, b_3, c_3\}$  and by  $\{a_3, a_1, x, b_3, b_2, c_2\}$  are  $Z_3$ 's. This implies that  $b_2$  and  $b_3$  are heavy in  $G'$ . Since there are two vertices  $a_1, x$  nonadjacent to  $b_2$  and  $b_3$ ,  $b_2$  and  $b_3$  have at least two common neighbors in  $G'$ . Let  $y$  be a common neighbor of  $b_2$  and  $b_3$  in  $G'$  other than  $b_1$ . Then  $y \in V(B)$ , and the subgraphs of  $G'$  induced by  $\{b_2, b_1, y, a_2, a_3, c_3\}$  is a  $Z_3$ . Thus  $a_2$  is heavy in  $G'$ . By Lemma 1 (3),  $a_2$  and  $b_3$  are associated, a contradiction. Thus we conclude that  $|V(A)| = 3$ , and similarly,  $|V(B)| = 3$ .

Suppose that  $|V(C_i)| \geq 4$  for  $l_i = T$ . We assume up to symmetry that  $|V(C_2)| \geq 4$ . Let  $x$  be a vertex in  $V(C_2) \setminus \{a_2, b_2, c_2\}$ . Then the subgraph of  $G'$  induced by  $\{a_2, c_2, x, a_3, b_3, b_1\}$  is a  $Z_3$ , implying that  $a_3$  is heavy in  $G$ . If  $l_1 = T$ , then the subgraph of  $G'$  induced by  $\{b_2, c_2, x, b_1, a_1, a_3\}$  is a  $Z_3$ ; if  $l_1 = 3$ , then the subgraph of  $G'$  induced by  $\{b_2, c_2, x, b_1, c_1, a_1\}$  is a  $Z_3$ . In any case, we have  $b_1$  is heavy in  $G'$ . But  $a_3$  and  $b_1$  are dissociated in  $G$ , a contradiction.

Suppose that  $l_1 = 3$  and  $|V(C_1^1)| \geq 3$ . Let  $x$  be a vertex in  $V(C_1^1) \setminus \{a_1, c_1^1\}$ . Then the subgraphs of  $G'$  induced by  $\{a_1, c_1^1, x, a_2, b_2, b_3\}$  and by  $\{c_1^1, a_1, x, b_1, b_2, c_2\}$  are  $Z_3$ 's. This implies that  $a_2$  and  $b_1$  are heavy in  $G'$ . But  $a_2$  and  $b_1$  are dissociated, a contradiction. Thus we conclude that  $|V(C_1^1)| = 2$ , and similarly,  $|V(C_1^2)| = 2$ .  $\square$   $\square$

In the following, we set  $S = \{v \in V(G') : N_{G'}(v) \cap V(H) \neq \emptyset\}$ .

**Claim 3.**  $l_1 = 3$ , and for  $x \in S$ ,  $xc_2, xc_3 \in E(G')$ .

*Proof.* By Claim 2, all the neighbors of  $a_1, a_2, a_3, b_1, b_2, b_3$  and  $c_1^1$  (if  $l_1 = 3$ ) are in  $H$ . Note that  $G'$  has at least 10 vertices. The vertices  $a_1, a_2, a_3, b_1, b_2, b_3$  and  $c_1^1$  (if  $l_1 = 3$ ) are not heavy in  $G'$ .

Let  $x$  be a vertex in  $S$ . Suppose that  $l_1 = T$ . Note that  $x$  cannot be adjacent to all the three vertices  $c_1, c_2, c_3$ . We assume up to symmetry that  $xc_1 \in E(G')$  and  $xc_2 \notin E(G')$ . Then the subgraph of  $G'$  induced by  $\{a_2, b_2, c_2, a_1, c_1, x\}$  is a  $Z_3$ , implying  $a_1$  is heavy in  $G'$ , a contradiction. Thus we conclude that  $l_1 = 3$ .

Suppose that one edge of  $xc_2, xc_3$  is not in  $E(G')$ , say  $xc_2 \notin E(G')$ . Then the subgraph of  $G'$  induced by  $\{a_2, b_2, c_2, a_3, c_3, x\}$  is a  $Z_3$ , implying  $a_3$  is heavy in  $G'$ , a contradiction. Thus we conclude that  $xc_2, xc_3 \in E(G')$ .  $\square$   $\square$

Let  $x$  be a vertex in  $S$ . By Claim 3,  $xc_2, xc_3 \in E(G')$ . If  $G'$  has only ten vertices, then  $C = a_1a_2a_3c_3xc_2b_2b_3b_1c_1a_1$  is a Hamilton cycle of  $G'$ , a contradiction. Suppose now that  $G'$  has an eleventh vertex. Since  $G'$  is 2-connected, let  $x'$  be a vertex in  $S \setminus \{x\}$ . By Claim 3,  $x'c_2, x'c_3 \in E(G')$ . Thus  $xx' \in E(G')$ . Note that  $N_{G'}(x)$  is neither a clique nor a disjoint union of two cliques of  $G'$ . This implies that  $x$  is an  $o$ -eligible vertex of  $G'$ , a contradiction.

The proof is complete.  $\square$

## References

- [1] P. Bedrossian, Forbidden Subgraph and Minimum Degree Conditons for Hamiltonicity, Ph.D. Thesis, Memphis State University (1991)
- [2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan London and Elsevier, New York (1976)
- [3] J. Brousek, Minimal 2-connected non-hamiltonian claw-free graphs, Discrete Math. 191, 57–64 (1998)
- [4] R. Čada, Degree conditions on induced claws, Discrete Math. 308, 5622–5631 (2008)
- [5] G. Chen, B. Wei, X. Zhang, Degree-light-free graphs and hamiltonian cycles, Graphs Combin. 17, 409–434 (2001)
- [6] R.J. Faudree, R.J. Gould, Characterizing forbidden pairs for hamiltonian properties, Discrete Math. 173, 45–60 (1997)
- [7] R.J. Faudree, R.J. Gould, Z. Ryjáček, I. Schiermeyer, Forbidden subgraphs and pancyclicity, Congress Numer. 109, 13–32 (1995)
- [8] B. Li, Z. Ryjáček, Y. Wang, S. Zhang, Pairs of heavy subgraphs for Hamiltonicity of 2-connected graphs, SIAM J. Discrete Math. 26, 1088–1103 (2012)
- [9] B. Ning, S. Zhang, Ore- and Fan-type heavy subgraphs for Hamiltonicity of 2-connected graphs, Discrete Math. 313, 1715–1725 (2013)
- [10] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B 70, 217–224 (1997)